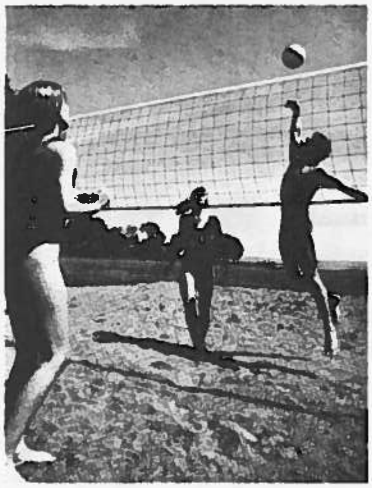
5.5 Factor Quadratic Expressions of the Form $ax^2 + bx + c$



The Ontario Summer Games are held every two years in even-numbered years to provide sports competition for youth between the ages of 11 and 22. At the Games, approximately 2500 athletes from across the province compete in 19 sports.

Beach volleyball is one of the sports on the Games program. It is played by two teams of two players on a sand court with area given by $3x^2 + 10x + 3$. Algebraic expressions for the dimensions of the court can be found by factoring the trinomial expression.

Investigate

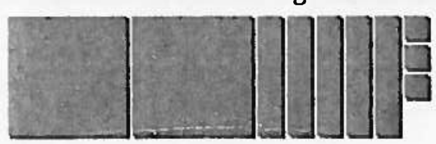
Tools

- algebra tiles

How can you use a model to factor quadratic expressions of the form $ax^2 + bx + c$?

1. Use algebra tiles to form a rectangle to model the product $(2x + 1)(x + 2)$.

2. Arrange the algebra tiles representing the trinomial $2x^2 + 5x + 3$ to form a rectangle. Identify the binomials that represent the length and width of the rectangle.



3. Repeat step 2 for each trinomial.
   a) $3x^2 + 5x + 2$ **b)** $4x^2 + 8x + 3$

4. Each trinomial represents the area of a rectangle. Draw diagrams and identify the binomials that represent the length and width of the rectangle.
   a) $2x^2 + 5x + 2$ **b)** $5x^2 + 8x + 3$

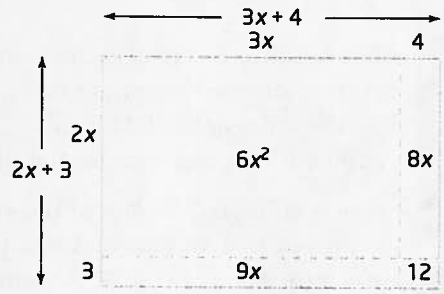
5. **Reflect** Describe how to use algebra tiles to factor a quadratic trinomial of the form $ax^2 + bx + c$.

6. **Reflect** Can you see a way to factor trinomials of the form $ax^2 + bx + c$ without using algebra tiles? If so, describe it.

When you expand two binomials, you add the two middle terms.

$$(2x + 3)(3x + 4) = 6x^2 + 8x + 9x + 12$$
$$= 6x^2 + 17x + 12$$

Notice the following patterns.
$8 + 9 = 17$ and $8 \times 9 = 6 \times 12$



You can use these patterns and the method of factoring by grouping to factor trinomials of the form $ax^2 + bx + c$. Work in reverse by replacing the middle term with two terms whose integer coefficients have a product of $a \times c$ and a sum of b.

Example 1 Break up the Middle Term

Factor, if possible.

a) $3x^2 + 8x + 4$

b) $3x^2 + 2x + 4$

c) $6x^2 - 5x + 1$

Solution

a) For $3x^2 + 8x + 4$, $a = 3$, $b = 8$, and $c = 4$.
Use a table to find two integers whose product is 3×4, or 12, and whose sum is 8. In order to have a positive product and a positive sum, both integers must be positive.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

Since the integers 2 and 6 satisfy this product and sum, break up $8x$ into $2x + 6x$. Then, factor by grouping.

$$\begin{aligned} & 3x^2 + 8x + 4 \\ &= 3x^2 + 2x + 6x + 4 \\ &= (3x^2 + 2x) + (6x + 4) \\ &= x(3x + 2) + 2(3x + 2) \\ &= (3x + 2)(x + 2) \end{aligned}$$

Literacy Connections

Since *break up* and *decompose* mean the same thing, the method of breaking up the middle term is sometimes referred to as the decomposition method.

b) For $3x^2 + 2x + 4$, $a = 3$, $b = 2$, and $c = 4$.
Since there is no pair of integers that satisfy these conditions, $3x^2 + 2x + 4$ is not factorable over the integers.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

I need to find two integers whose product is 3×4, or 12, and whose sum is 2. Since the product and the sum are positive, I need two positive integers.

c) For $6x^2 - 5x + 1$, $a = 6$, $b = -5$, and $c = 1$.
Since the integers -2 and -3 satisfy this product and sum, break up $-5x$ into $-2x - 3x$. Then, factor by grouping.

$$
\begin{aligned}
&6x^2 - 5x + 1 \\
&= 6x^2 - 2x - 3x + 1 \\
&= (6x^2 - 2x) + (-3x + 1) \\
&= 2x(3x - 1) - 1(3x - 1) \\
&= (3x - 1)(2x - 1)
\end{aligned}
$$

Factors of 6	Product	Sum
−1, −6	6	−7
−2, −3	6	−5

I need to find two integers whose product is 6×1, or 6, and whose sum is −5. Since the product is positive and the sum is negative, I need two negative integers.

Example 2 Trinomials With Two Variables

Factor $10x^2 - 3xy - 4y^2$.

Solution

For $10x^2 - 3xy - 4y^2$, $a = 10$, $b = -3$, and $c = -4$.

I need to find two integers whose product is $10 \times (-4)$, or −40, and whose sum is −3. The integers 5 and −8 work.

Factors of −40	Product	Sum
1, −40	−40	−39
2, −20	−40	−18
4, −10	−40	−6
5, −8	−40	−3
−1, 40	−40	39
−2, 20	−40	18
−4, 10	−40	6
−5, 8	−40	3

$$
\begin{aligned}
&10x^2 - 3xy - 4y^2 \\
&= 10x^2 + 5xy - 8xy - 4y^2 \\
&= (10x^2 + 5xy) + (-8xy - 4y^2) \\
&= 5x(2x + y) - 4y(2x + y) \\
&= (2x + y)(5x - 4y)
\end{aligned}
$$

Break up $-3xy$ into $5xy - 8xy$.
Factor by grouping.

Example 3 Remove a Common Factor

Factor $16x^2 + 26x - 12$.

Solution

First, remove the greatest common factor (GCF), and then proceed as before.

The GCF of the polynomial $16x^2 + 26x - 12$ is 2.

$$\begin{aligned} &16x^2 + 26x - 12 \\ &= 2(8x^2 + 13x - 6) \end{aligned}$$

To factor $8x^2 + 13x - 6$, I need to find two integers whose product is $8 \times (-6)$, or -48, and whose sum is 13. The integers -3 and 16 work.

Factors of −48	Product	Sum
−1, 48	−48	47
−2, 24	−48	22
−4, 12	−48	8
−6, 8	−48	2
1, −48	−48	−47
2, −24	−48	−22
3, −16	−48	−13
4, −12	−48	−8
6, −8	−48	−2

$$\begin{aligned} 16x^2 + 26x - 12 &= 2(8x^2 + 13x - 6) \\ &= 2(8x^2 - 3x + 16x - 6) && \text{Break up } 13x \text{ into } -3x + 16x. \\ &= 2[(8x^2 - 3x) + (16x - 6)] && \text{Factor by grouping.} \\ &= 2[x(8x - 3) + 2(8x - 3)] \\ &= 2[(8x - 3)(x + 2)] \\ &= 2(8x - 3)(x + 2) \end{aligned}$$

Key Concepts

Always look for a common factor first when factoring a trinomial.

To factor $ax^2 + bx + c$, find two integers whose product is $a \times c$ and whose sum is b. Then, break up the middle term and factor by grouping.

Not all quadratic expressions of the form $ax^2 + bx + c$ can be factored over the integers.

Communicate Your Understanding

When you use algebra tiles to factor a trinomial, why do you need to be able to form a rectangle with the tiles?

When factored, $2x^2 + 9x + 9$ can be written as $(2x + 3)(x + 3)$. Can it also be written as $(x + 3)(2x + 3)$? Justify your answer using words and a diagram.

Describe how you would factor $5x^2 + 18x + 9$.